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Approximate solutions of predictive relativistic mechanics for the gravitational interaction

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Abstract. In this paper and following a method developed by Bel, Salas and Sánchez-Ron the equations of predictive relativistic mechanics are solved for the gravitational interaction of two structureless point particles, up to the second order in the coupling constant $g \equiv Gm_1m_2$. The equations of motion for the 'static' case are explicitly given as well as the value for the perihelion shift of a planet in the corresponding Kepler problem.

1. Predictive relativistic mechanics

Predictive relativistic mechanics (PRM) deals with the problem of formulating an action-at-a-distance relativistic theory of structureless point particles which interact among themselves. Up to now, in spite of the fact that PRM has proved to be quite successful and seems to be one of the few (possibly the only) consistent models for a dynamics compatible with the postulates of special relativity, it has only been applied to the electromagnetic (Bel *et al* 1973) and the short range scalar (Bel and Martin 1974) interactions, not to the gravitational interaction. The purpose of this paper is to initiate the treatment of gravitation within the framework of PRM.

In this section we shall briefly review those definitions and results of PRM which we shall need later. For a more complete discussion the reader is referred to Droz-Vincent (1970) and Bel (1970, 1971).

Let us consider the Minkowskian space–time M_4 (signature $\eta_{\alpha\beta} = +2$). A Poincaré-invariant predictive system is described by equations of motion which are autonomous ordinary second-order differential equations (we shall make use of the manifestly covariant formalism instead of the time-symmetric one):

$$\frac{\mathrm{d}x_a^\alpha}{\mathrm{d}\tau} = u_a^\alpha, \qquad \frac{\mathrm{d}u_a^\alpha}{\mathrm{d}\tau} = \xi_a^\alpha(x_b^\beta, u_c^\gamma) \tag{1.1}$$

where $\alpha, \beta, \gamma, \ldots = 0, 1, 2, 3$; $a, b, c, \ldots = 1, 2, \ldots, N$, N representing the number of particles. This verifies firstly

$$\psi_a^\alpha(L_\mu^{\beta'}(x_{0b}^\mu - A^\mu), L_\mu^{\gamma'} u_{0c}^\mu; \tau) = L_\mu^{\alpha'}[\psi_a^\mu(x_{0b}^\beta, u_{0c}^\gamma; \tau) - A^\mu], \qquad (\alpha = \alpha'), \tag{1.2}$$

† This work was initiated during the author's stay at the Department of Mathematics, King's College, London, supported by a European Space Research Organisation research studentship.

for each Poincaré transformation $(L^{\alpha'}_{\beta}, A^{\gamma})$. ψ^{α}_{a} is the general solution of system (1.1), i.e.

$$x^{\alpha}_{a} = \psi^{\alpha}_{a}(x^{\beta}_{0b}, u^{\gamma}_{0c}; \tau), \qquad u^{\alpha}_{a} = \frac{d\psi^{\alpha}_{a}}{d\tau}(x^{\beta}_{0b}, u^{\gamma}_{0c}; \tau);$$

and secondly

$$u^{\beta}_{a}\frac{\partial \xi^{\gamma}_{b}}{\partial x^{\beta}_{a}} + \xi^{\beta}_{a}\frac{\partial \xi^{\gamma}_{b}}{\partial u^{\beta}_{a}} = 0 \tag{1.3}$$

($a \neq b$, no summation over a) and

$$\xi^{\beta}_{a} u_{a\beta} = 0. \tag{1.4}$$

Obviously equation (1.4) implies that $u^{\beta}_{a}u_{a\beta}$ is constant along each trajectory. (In the following we shall consider only the two-body problem ($N=2$), that is $a, b = 1, 2$, $a \neq b$).

The exigence of invariance under the orthochronous Poincaré group permits us to express the ξ in the following way:

$$\xi^{\beta}_{a} = (-1)^{b} a_{a} x^{\beta} + b_{aa} u^{\beta}_{a} + b_{ab} u^{\beta}_{b}, \tag{1.5}$$

where $x^{\beta} \equiv (-1)^{b}(x^{\beta}_{a} - x^{\beta}_{b})$, and the coefficients a_a, b_{aa}, b_{ab} are functions of the four scalar variables:

$$x^{2} \equiv x^{\alpha}x_{\alpha}, \qquad (xu_{a}) \equiv x^{\beta}_{a}u_{a\beta}, \qquad k \equiv -u^{\beta}_{a}u_{b\beta}, \tag{1.6}$$

(unless otherwise stated, the units will be so chosen that the speed of light in vacuum will be $c = 1$). Since u^{β}_{a} will be taken as future-oriented ($u^{0}_{a} > 0$), timelike vectors, $u_{a} \equiv +(-u^{\beta}_{a}u_{a\beta})^{1/2}$, are real and k is positive.

In a preceding paper, Bel *et al* (1973) have shown how to solve equations (1.3)–(1.5) assuming that the unknowns could be expressed as power series in a coupling constant g. In that paper this method was applied to the electromagnetic interaction taking $g \equiv e_1 e_2$ (e_a is equivalent to the charge of particle a) and using the formula of Liénard and Wiechert as a boundary condition. Afterwards Bel and Martin (1974) considered the short range scalar interaction within the same framework. In the present paper we apply the general method developed by Bel *et al* (1973) to the gravitational interaction making use also of Bel and Martin's results (Bel and Martin 1974) conveniently particularized.

2. The Galilean principle of equivalence

It is today widely accepted that the first requisite which a theory must fulfil in order to describe the gravitational interaction is to incorporate the Galilean (or weak) principle of equivalence, namely: the ratio of inertial to gravitational mass is the same for all objects (all bodies fall equally fast). This requisite is firmly grounded on the work of Roll *et al* (1964) and Braginsky and Panov (see Braginsky 1974). The value today accepted for the relative difference of the ratio of the inertial to the gravitational mass for different bodies is $(-0{\cdot}3 \pm 0{\cdot}9)\ 10^{-12}$. This means in fact that the Galilean principle of equivalence is one of the best experimentally supported results in all physics.

Considering PRM, the equality between inertial and gravitational masses can be incorporated in the same way as in Newtonian mechanics, i.e. as a separate exigence.

This is due to the fact that, as we saw in § 1, the equations of motion in PRM are of the same type as the corresponding ones in Newtonian mechanics. Furthermore, due to constraints (1.3) and (1.4), the functions $\xi^\alpha_a(x^\beta_b, u^\gamma_c)$ depend on $6N$ initial conditions; that is, given this number of initial conditions the world lines of the particles are completely determined, as happens in Newtonian mechanics.

3. The two-body problem: recurrent method to obtain the accelerations†

In order to determine the functions ξ^α_a for $N=2$ it is assumed that they can be expanded into power series with a coupling constant g:

$$\xi^\alpha_a = \sum_{n=1}^{\infty} g^n \xi^{(n)\alpha}_a. \tag{3.1}$$

Introducing these expansions into (1.5) we have

$$\xi^{(n)\alpha}_a = (-1)^b a^{(n)}_a x^\alpha + b^{(n)}_{aa} u^\alpha_a + b^{(n)}_{ab} u^\alpha_b. \tag{3.2}$$

Solving order by order equations (1.3) and (1.4), the general solution can be expressed in two different forms ($\epsilon = \pm 1$):

$$a^{(n)}_a(\epsilon) = -\int_{(-1)^b \epsilon r_b}^{(xu_b)} A^{(n)}_a(r_b, s_b, k, y)\, \mathrm{d}y + a^{(n)*}_a(r_b, s_b, k; \epsilon) \tag{3.3}$$

$$b^{(n)}_{aa}(\epsilon) = -\int_{(-1)^b \epsilon r_b}^{(xu_b)} B^{(n)}_{aa}(r_b, s_b, k, y)\, \mathrm{d}y + b^{(n)*}_{aa}(r_b, s_b, k\ ; \epsilon) \tag{3.4}$$

$$b^{(n)}_{ab}(\epsilon) = k^{-1}[(-1)^b (xu_a) a^{(n)}_a(\epsilon) - b^{(n)}_{aa}(\epsilon)]. \tag{3.5}$$

The integrands $A^{(n)}_a$ and $B^{(n)}_{aa}$ are defined by

$$(-1)^b \sum_{p+q=n} \xi^{(p)\beta}_b \frac{\partial \xi^{(q)\alpha}_a}{\partial u^\beta_b} = (-1)^b A^{(n)}_a x^\alpha + B^{(n)}_{aa} u^\alpha_a + B^{(n)}_{ab} u^\alpha_b \tag{3.6}$$

with $A^{(1)}_a = B^{(1)}_{aa} = B^{(1)}_{ab} = 0$.

$A^{(n)}_a$ and $B^{(n)}_{aa}$ are functions of the following scalars

$$\{r_b \equiv [x^2 + (xu_b)^2]^{1/2},\ s_b \equiv (xu_a) - k(xu_b),\ k,\ (xu_b)\} \tag{3.7}$$

all, except the last one, being kept constant during the integrations in (3.3) and (3.4).

4. Gravitational interaction

It is well known (Anderson 1967) that, in order to describe the gravitational interaction of massive bodies, one of the possibilities is to represent the gravitational field by a scalar field $\phi(x)$ within the framework of special relativity and to replace the inhomogeneous Laplacian equation by the inhomogeneous wave equation

$$\Box \phi(x) = 4\pi G \rho(x)$$

† Here the recurrent method is only sketched. For a complete discussion the reader is referred to Bel *et al* (1973), Salas and Sánchez-Ron (1974), and Bel and Martin (1974).

where

$$\rho(x) = \sum_a m_a \int \delta^4(x - x_a)(u_a^\beta u_{a\beta})^{1/2} \, d\lambda_a.$$

Such a theory is equivalent to the meson field theory of field equation

$$(\Box + \alpha^2)\phi(x) = 4\pi\rho(x)$$

where

$$\rho(x) = \sum_a e_a \int \delta^4(x - x_a)(u_a^\beta u_{a\beta})^{1/2} \, d\lambda_a,$$

with $\alpha = 0$ and the mesonic charge e_a replaced by Gm_a. Therefore we can use the results corresponding to the mesonic interaction in PRM (Bel and Martin 1974) conveniently particularized and with an appropriate coupling constant.

In that sense let us now consider an isolated system of two point-like structureless particles interacting gravitationally. Let m_a be the mass of particle $\boldsymbol{a}$ and $\psi_a^\alpha(\tau_a)$ the parametrized equations of the world-line of particle $\boldsymbol{a}$, τ_a being the corresponding proper time. The retarded or advanced potentials ($\epsilon = -1$ or $\epsilon = +1$) due to particle $\boldsymbol{b}$ at a given point of the world-line of particle $\boldsymbol{a}$ with which $\boldsymbol{b}$ interacts gravitationally ($\alpha = 0$) are (see for example Havas 1952)

$$\phi_\epsilon(x_a^\alpha) = -2Gm_b \int_{-\infty}^{+\infty} \theta(\epsilon l_b^0)\delta(-l_b^\beta l_{b\beta}) \, d\tau_b \tag{4.1}$$

where θ is the Heaviside step function, δ the Dirac distribution and $l_b^\alpha \equiv \psi_b^\alpha(\tau_b) - x_a^\alpha$. Let u_b^α be the unit four-velocity at the point x_b^α; by integrating (4.1) we have

$$\phi_\epsilon(x_a^\alpha) = (-1)^a \epsilon G m_b (x u_b)^{-1} \tag{4.2}$$

where it is assumed that

$$x^\alpha x_\alpha = 0, \qquad \operatorname{sgn}(x^0) = (-1)^a \epsilon. \tag{4.3}$$

By taking the derivative of equation (4.2) with respect to x_a^β and taking into account conditions (4.3) we obtain

$$\frac{\partial\phi_\epsilon}{\partial x_a^\beta}(x_a^\alpha) = \epsilon G m_b (x u_b)^{-2}\{u_{b\beta} + (x u_b)^{-1}[1 + (-1)^b (x\xi_b)]x_\beta\} \tag{4.4}$$

where again the constraints (4.3) must be assumed.

Following a similar approach to the one of Bel and Martin (1974) we shall consider the following equation of motion

$$(\delta_\mu^\sigma + u_a^\sigma u_{a\mu})\frac{\partial\phi_\epsilon}{\partial x_a^\sigma}(x_a^\nu) + [1 + \chi\phi_\epsilon(x_a^\nu)]\frac{du_{a\mu}}{d\tau_a} = 0 \tag{4.5}$$

where χ is an arbitrary constant. (In the equation they consider, Bel and Martin give the values 0 or 1 to the constant χ in order to deal with the two scalar theories which have been most often treated in the literature; however this is not necessary and for the moment we shall keep an arbitrary χ).

Substituting equations (4.2) into equation (4.5) we obtain the following equation which will play the same role of boundary condition as the Liénard–Wiechert formula does in the electromagnetic interaction

$$\xi_a^\alpha = -gm_a^{-1}(\eta^{\alpha\beta} + u_a^\alpha u_a^\beta)(xu_b)^{-2}[\epsilon + \chi(-1)^a gm_a^{-1}(xu_b)^{-1}] \times\{u_{b\beta} + (xu_b)^{-1}[1+(-1)^b(x\xi_b)]x_\beta\}, \tag{4.6}$$

where $g \equiv Gm_am_b$.

The first and second order terms, in the coupling constant g, of equations (4.6) are

$$\xi_a^{(1)\alpha} = -\epsilon m_a^{-1}(\eta^{\alpha\beta} + u_a^\alpha u_a^\beta)(xu_b)^{-2}[u_{b\beta} + (xu_b)^{-1}x_\beta] \tag{4.7}$$

$$\xi_a^{(2)\alpha} = (-1)^a m_a^{-2}(xu_b)^{-3}(\eta^{\alpha\beta} + u_a^\alpha u_a^\beta)\{\chi[u_{b\beta} + (xu_b)^{-1}x_\beta] + \epsilon m_a(x\xi_b^{(1)})x_\beta\}. \tag{4.8}$$

Using the notation $f^\alpha(\epsilon;\mp) \equiv$ value of $f^\alpha(x_b^\beta, u_c^\gamma;\epsilon)$ when the vector $(-1)^b x^\alpha$ is null and future-oriented $(-)$ or null and past-oriented $(+)$ we have

$$\xi_a^{(1)\alpha}(\epsilon;\epsilon) = -\epsilon m_a^{-1}(\eta^{\alpha\beta} + u_a^\alpha u_a^\beta)(xu_b)^{-2}[u_{b\beta} + (xu_b)^{-1}x_\beta], \tag{4.9}$$

where it is assumed that $x^\alpha x_\alpha = 0$. From equation (3.2) we get

$$a_a^{(1)}(\epsilon;\epsilon) = (-1)^a \epsilon m_a^{-1}(xu_b)^{-3} \tag{4.10}$$

$$b_{aa}^{(1)}(\epsilon;\epsilon) = -\epsilon m_a^{-1}(xu_b)^{-3}s_b, \tag{4.11}$$

where (3.7) has been used.

But from equations (3.3)–(3.6) we have

$$\begin{aligned} a_a^{(1)}(\epsilon) &= a_a^{(1)*}(r_b, s_b, k;\epsilon) \\ b_{aa}^{(1)}(\epsilon) &= b_{aa}^{(1)*}(r_b, s_b, k;\epsilon), \end{aligned} \tag{4.12}$$

and assuming that $x^\alpha x_\alpha = 0$, $x^0 = (-1)^a\epsilon|x^0|$, we use equations (3.7) to get:

$$(xu_a) = s_b + (-1)^b\epsilon k r_b, \qquad (xu_b) = (-1)^b \epsilon r_b. \tag{4.13}$$

Substituting equations (4.13) into expressions (4.10) and (4.11) we obtain

$$a_a^{(1)}(\epsilon) = -m_a^{-1}r_b^{-3} \tag{4.14}$$

$$b_{aa}^{(1)}(\epsilon) = (-1)^a m_a^{-1} r_b^{-3} s_b; \tag{4.15}$$

and using equation (3.5)

$$b_{ab}^{(1)}(\epsilon) = (-1)^a m_a^{-1}(xu_b)r_b^{-3}. \tag{4.16}$$

From expressions (4.14)–(4.16) we get

$$\xi_a^{(1)\alpha}(x_b^\beta, u_c^\gamma;\epsilon) = (-1)^a m_a^{-1} r_b^{-3}[x^\alpha + s_b u_a^\alpha + (xu_b)u_b^\alpha]. \tag{4.17}$$

Let us now compute $\xi_a^{(2)\alpha}(x_b^\beta, u_c^\gamma;\epsilon)$. The first step is to calculate $\xi_a^{(2)\alpha}(\epsilon;\epsilon)$ using equations (4.8). Let us start by computing $(x\xi_b^{(1)}(\epsilon;-\epsilon))$,

$$(x\xi_b^{(1)}(\epsilon;-\epsilon)) = \epsilon m_b^{-1}[s_b + k(xu_b)]^{-3}[s_b^2 + (xu_b)^2 + ks_b(xu_b)]. \tag{4.18}$$

Substituting (4.18) into (4.8) we obtain the coefficients of $\xi_a^{(2)\alpha}(\epsilon;\epsilon)$ in the expression

$$\xi_a^{(2)\alpha}(\epsilon;\epsilon) = (-1)^b a_a^{(2)}(\epsilon;\epsilon)x^\alpha + b_{aa}^{(2)}(\epsilon;\epsilon)u_a^\alpha + b_{ab}^{(2)}(\epsilon;\epsilon)u_b^\alpha.$$

The explicit expressions of the coefficients are

$$a_a^{(2)}(\epsilon;\epsilon) = -\chi m_a^{-2}(xu_b)^{-4} - m_a^{-1}m_b^{-1}(xu_b)^{-3}[s_b + k(xu_b)]^{-3}[s_b^2 + (xu_b)^2 + ks_b(xu_b)] \tag{4.19}$$

$$b_{aa}^{(2)}(\epsilon;\epsilon)=(-1)^a(xu_b)^{-3}\{m_a^{-2}[2+s_b(xu_b)^{-1}]+m_a^{-1}m_b^{-1}[s_b+(xu_b)] \times[s_b+k(xu_b)]^{-3}[s_b^2+(xu_b)^2+ks_b(xu_b)]\}; \tag{4.20}$$

$b_{ab}^{(2)}(\epsilon;\epsilon)$ will be obtained substituting (4.19) and (4.20) into (3.5).

Using expressions (3.3), (3.6), (4.17) and (3.7)† we obtain

$$a_a^{(2)}(\epsilon)=-m_a^{-1}m_b^{-1}r_b^{-3}\int_{(-1)^b\epsilon r_b}^{(xu_b)} y[3r_b^{-2}(r_b^2+s_b^2+ks_by)-1] \times[r_b^2+s_b^2+2ks_by+(k^2-1)y^2]^{-3/2}\,\mathrm{d}y+a_a^{(2)*}(r_b,s_b,k;\epsilon); \tag{4.21}$$

a similar expression can be obtain for $b_{aa}^{(2)}$ by using (3.4) instead of (3.3). The integrals which appear in these expressions can be easily calculated but the results obtained are rather cumbersome.

From (3.3) and (3.4) it is obvious that

$$a_a^{(n)}(\epsilon;\epsilon)=a_a^{(n)*}(\epsilon;\epsilon),\qquad b_{aa}^{(n)}(\epsilon;\epsilon)=b_{aa}^{(n)*}(\epsilon;\epsilon), \tag{4.22}$$

and therefore expressions (4.19) and (4.20) determine $a_a^{(2)*}(\epsilon,\epsilon)$ and $b_{aa}^{(2)*}(\epsilon,\epsilon)$ respectively. The functions $a_a^{(2)*}$ and $b_{aa}^{(2)*}$ can now be obtained just by substituting (xu_b) by $(-1)^b\epsilon r_b$ in (4.19) and (4.20) in accordance with equations (4.13).

Finally to obtain $b_{ab}^{(2)}(\epsilon)$ we just have to use the preceding results together with (3.5).

5. The 'static' gravitational interaction

We shall now use the preceding results in order to obtain a relativistic correction to the 'static' gravitational interaction given by Newton's law of gravitation. To do this we therefore need to know, up to the second order in the coupling constant $g=Gm_am_b$, the three-accelerations $\overset{i}{\mu}_a$ $(i,j,\ldots=1,2,3)$ when both particles have zero velocities *at a given instant.* Under these conditions we have

$$r_b\equiv r\equiv(x^ix_i)^{1/2},\qquad s_b=0,\qquad k=1,\qquad (xu_b)=0 \tag{5.1}$$

(in the considered instant only), and the three-acceleration of particle ***a*** is given by (see for example Salas and Sánchez-Ron 1974)

$$\overset{i}{\mu}_a=(-1)^b(g\hat{a}_a^{(1)}+g^2\hat{a}_a^{(2)}+\ldots)x^i, \tag{5.2}$$

where $\hat{a}_a^{(n)}$ is the value of $a_a^{(n)}$ when its arguments take the values (5.1).

Introducing conditions (5.1) into equation (4.14) we trivially obtain

$$\hat{a}_a^{(1)}=-m_a^{-1}r^{-3}. \tag{5.3}$$

Let us now calculate $\hat{a}_a^{(2)}$. Substituting conditions (5.1) into expression (4.21) we get

$$\hat{a}_a^{(2)}=m_a^{-1}m_b^{-1}r^{-4}+a_a^{(2)*}(r,0,1,\epsilon) \tag{5.4}$$

where

$$a_a^{(2)*}(r,0,1,\epsilon)=-(\chi m_a^{-2}+m_a^{-1}m_b^{-1})r^{-4}; \tag{5.5}$$

that is,

$$\hat{a}_a^{(2)}=-\chi m_a^{-2}r^{-4}. \tag{5.6}$$

† The following relation is also needed

$$r_a^2=r_b^2+s_b^2+2ks_b(xu_b)+(k^2-1)(xu_b)^2.$$

It is worth noting that, to these orders, $a_a^{(1)}$ as well as $a_a^{(2)}$ are independent of ϵ, something which most probably will not happen for higher orders (see Salas and Sánchez-Ron 1974).

Substituting (5.3) and (5.6) into (5.2) we finally obtain

$$\mu_a^i = (-1)^a \frac{Gm_b}{r^3}\left(1+\chi\frac{Gm_b}{r}\right)x^i. \tag{5.7}$$

As it is obvious from equation (5.7), for the $\chi = 0$ scalar theory we only get (to the order of approximation considered here) Newton's law of gravitation.

Restoring c (speed of light) in equation (5.7) we get

$$\mu_a^i = (-1)^a \frac{Gm_b}{r^3}\left(1+\chi\frac{Gm_b}{c^2 r}\right)x^i. \tag{5.8}$$

Solving this system of six equations we should have a relativistic solution for the motion of two gravitating particles. In this paper and in order to compare our results with the classical astronomical tests we shall deal with the Kepler problem where one of the two bodies has a much larger mass than the other (one mass moving in a fixed field). Let us suppose for instance that $m_2 \gg m_1$. We can then write

$$\frac{\mu_2^i}{\mu_1^i} = -\left(\frac{m_1}{m_2}\right)\left(\frac{c^2 r+\chi Gm_1}{c^2 r+\chi Gm_2}\right)$$

which obviously tends to zero when $m_2 \gg m_1$; that is $\mu_2^i \simeq 0$. Therefore and without loss of generality we have $x_2^i = 0$ (using the fact that at a given instant $u_2^i = 0$). As a consequence of all these considerations (5.8) takes the following form

$$\ddot{x}^i = -\frac{Gm}{r^3}\left(1+\chi\frac{Gm}{c^2 r}\right)x^i, \tag{5.9}$$

where $x^i \equiv x_1^i$, $\ddot{x}^i \equiv \mathrm{d}^2x^i/\mathrm{d}t^2$, $m \equiv m_2$.

Since the force in (5.9) is central we have a planar motion (i.e. $x^3 = 0$). Equations (5.9) then reduce to

$$\begin{aligned}\ddot{x} &= -\frac{Gm}{r^3}\left(1+\chi\frac{Gm}{c^2 r}\right)x\\ \ddot{y} &= -\frac{Gm}{r^3}\left(1+\chi\frac{Gm}{c^2 r}\right)y,\end{aligned} \tag{5.10}$$

where $x \equiv x^1$, $y \equiv x^2$, $r \equiv (x^2+y^2)^{1/2}$. It is convenient to replace (x, y) with polar coordinates (r, φ) defined as usual by

$$x = r\cos\varphi, \qquad y = r\sin\varphi.$$

In this system equations (5.10) take the following form

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{Gm}{r^2}\left(1+\chi\frac{Gm}{c^2 r}\right) \tag{5.11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}) = 0. \tag{5.11b}$$

From (5.11*b*) we obtain immediately

$$r^2\dot{\varphi} = h \equiv \text{constant}. \tag{5.12}$$

For our purposes it is convenient to consider r as a function of φ instead of t. Making use of a new variable $u \equiv 1/r$ and substituting (5.12) in (5.11*a*) we finally arrive at the following equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{Gm}{h^2} + \chi\left(\frac{Gm}{ch}\right)^2 u \tag{5.13}$$

which differs from the corresponding equation in Newton's theory by the second term on the right-hand side.

It is interesting to note that Chazy (1928) considered almost fifty years ago, from a Newtonian perspective, a system of equations which contained equations (5.10). Making $\alpha = -\chi/2$, $\beta = \gamma = \lambda = 0$ in Chazy's equations (see Chazy 1928, p 106, equations (15)) we obtain (5.10). Also Sommerfeld's theory of the hydrogen fine structure for $\chi = 1$ leads to our results (see for example Sommerfeld 1923, 1952).

From equation (5.13) we obtain for our theory an *advance* (if $\chi > 0$) of the perihelion of

$$\delta\bar{\omega} = \frac{\chi\pi Gm}{c^2a(1-e^2)}$$

radians per revolution (a and e are essentially the semi-major axis and the eccentricity of the Kepler ellipse respectively). Therefore for $\chi = 1$ the result is a sixth of the value predicted by Einstein's general relativity.

Due to the fact that $\chi = 1$ corresponds to one of the classical scalar theories which give a *retardation* for the perihelion shift of a planet, our results seem to contradict this standard result for a scalar interaction, but it must be taken into account that we have made use of the 'scalar formula' (4.6) only as a boundary condition to obtain approximate solutions of the equations of PRM and as far as we can see there is no reason why, in this situation, we must regain the same results as given by scalar theory.

We can obtain Einstein's results for the advance of the perihelion by using as a boundary condition a scalar theory with $\chi = 6$. In that situation we shall have the following equation of motion instead of (4.5)

$$(\delta^\sigma_\mu + u_{a\mu}u^\sigma_a)\frac{\partial\phi_\epsilon}{\partial x^\sigma_a} + (1 + 6\phi_\epsilon)\frac{\mathrm{d}u_{a\mu}}{\mathrm{d}\tau_a} = 0. \tag{5.14}$$

The election of the perihelion shift of a planet as the only test by which to compare our results is motivated by the fact that it is today widely admitted that the perihelion shift is the most significant result for testing the purely gravidynamical aspects of a theory which tries to describe the gravitational field. To calculate, for example, the bending of light in a Schwarzschild field—usually the fatal test for standard scalar theories—it is necessary to develop a theory of interaction between the electromagnetic and the gravitational fields, a problem which is not yet solved within the scope of this paper. Moreover the method used here is particularly unsuitable for dealing with massless particles ($m = 0$) such as the photon, because in this case the algorithm of perturbations on the coupling constant GMm, ($M \equiv m_1$), breaks down. In fact, up to now, no equations of motion for massless particles have been developed in PRM.

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